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Analysis and Control of Bifurcation Phenomena in Aircraft Flight

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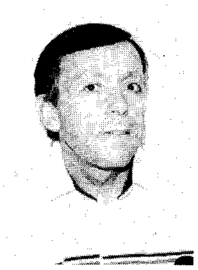
This paper addresses a theoretical framework for a unified methodology which allows analysis of nonlinear stability and efficient control of high-dimensional nonlinear plants modeling aircraft flight. It is shown that analysis of nonlinear transition phenomena (bifurcations) is central to revealing the limitation of robust control (i.e., an accurate estimate of the basin of stability). Omitting transition behavior causes over control and provides a very local stabilization. Analysis and control of bifurcations of aircraft flight are given in the spirit of the generalized normal forms method, which provides one with the nonreducible system that preserves stability characteristics of the initial plant. Stabilization of a plant's bifurcations is then given in terms of the resonance control methodology. Efficiency of the developed methodology is demonstrated by analyzing and controlling an unstable nonlinear plant relevant to the lateral dynamics of an aircraft. Whereas the initial plant is governed by a number of coupled nonlinear equations, the reduced system (the resonance normal form) turns out to be much easier to analyze and even integrable in many cases. Analysis of bifurcations of the resonance normal forms may shape efficient control actions which a pilot may undertake to ensure stability of an aircraft in a prescribed neighborhood of a trim condition and also can furnish a design of a flight's automatic control.

Introduction

A MODERN aircraft is designed to attain high speed and maneuverability. As an aircraft becomes faster and more maneuverable, however, it becomes more difficult to control at or near the extrema of the flight envelope. Although desirable performance of aircraft is often close to the extrema of the flight envelope, such regimes could be the cause of unstable behavior due to small disturbances. This is especially true in the case of high angle-of-attack (α) flights. Due to essential nonlinearity, unstable aircraft performance is often counterintuitive as well and may induce the pilot to attempt control actions which worsen the situation.

Previous research concentrated primarily on analysis/control of aircraft flight at a high angle of attack. The phenomena of a high incident flight such as stall, spin entry, flat and steep spin, nose slice, and wing rock have been interpreted as instability of steady-state solutions of governing nonlinear dynamic equations. Observed instabilities were described in terms of elementary bifurcations in a number of publications.[1–9] In particular, both Hopf and stationary bifurcations have been linked with unstable behavior for high α in several aircraft models.[1–4] The instabilities of lateral dynamics of a slender-wing aircraft are interpreted as Hopf bifurcations in Abed and Lee.[9] Nonlinear coupling between longitudinal and lateral dynamics as a source of another type of instability were observed in Young et al.[8] and other publications. In spite of only elementary bifurcations having been related to aircraft instabilities before (in this case the critical eigenvalues are a pure imaginary couple or a single zero), analysis of numerical simulations of certain aircraft models indicates that more complex bifurcations also may arise.



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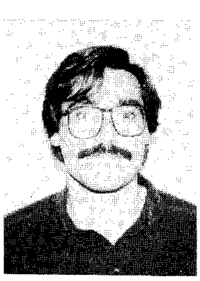

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One of the main objectives of dynamical analysis of aircraft is to facilitate design of a stabilizing control. Unfortunately, numerical approaches in nonlinear dynamics are not sufficient for such an aim. In the face of this problem, several techniques using feedback linearization were introduced for design of a stabilizing robust control.[10–12] It is known that feedback linearization fails for critical nonlinear plants undergoing bifurcations.

In many important practical cases stability can be characterized from the local point of view. Notice that all known types of aircraft instabilities can be described in terms of the local bifurcations of known steady-state solutions. Thus, to investigate and stabilize an aircraft's instabilities, we concentrate on designing an efficient feedback for the close-to-critical nonlinear systems, with a linear part having eigenvalues on or close to the imaginary axis.

Design of stabilizing feedback control for the critical dynamical systems (faced with bifurcations) has been studied by several authors. Existence of smooth feedback which asymptotically stabilizes an equilibrium point was studied by Brockett.[13] His consideration rests on Lyapunov functions. Aeyels[14–16] studied the same problem for a critical nonlinear system which exhibits Hopf bifurcation. His approach was based on the center manifold reduction to the standard Hopf bifurcation problem in two dimensions. Works of bifurcation control have been reported by Carroll and Mehra,[5] Kwatny et al.,[6] and Mehra et al.[7] Abed and Fu[17,18] obtain sufficient conditions for local smooth feedback stabilization in the critical cases when the linearized system has either a pair of pure imaginary eigenvalues or a single zero eigenvalue.

Although certain phenomena have been analyzed in isolation, there exists a clear need for a unified approach in order to systematically analyze and control an aircraft's unstable behavior.

Resonance Normal Forms Concept

We are concerned with developing an effective generic method for analysis of the local bifurcations of a broad class of multidimensional and multiparametric dynamical systems (such as aircraft dynamical models). A governing equation in the local nonlinear dynamics can be specified in the following form:

$$\dot{x} = A(\zeta)x + \epsilon f(x, t, \zeta, \epsilon) \tag{1}$$

where $x \in R^n$ is a vector of state variables, $0 < \epsilon < 1$ is a small parameter which corresponds to a size of a chosen neighborhood of equilibrium, A is a time invariant $n \times n$ matrix with simple spectrum which is represented in the diagonal form $f = \{f_1, \ldots, f_n\}$, and $\zeta \in R^m$ is a vector of system parameters. Stability transition of an equilibrium occurs if eigenvalues of A cross the imaginary axis. In such critical cases, the dominant linear terms compensate each other, and the transition behavior is determined by small nonlinear terms.

There are three basic techniques that have been used to approximate solutions of Eq. (1). The most primitive way to improve the linear approximation of Eq. (1) is to iterate solutions of the linearized equation successfully. The iterations work out easily for a very broad class of dynamical systems, but lead to rapidly divergent series in all practically interesting situations. Divergence is an intrinsic property of such kinds of series which occurs due to so-called resonance components in f.

The averaging method is considered as one of the deepest techniques for local nonlinear analysis. However, the use of averaging requires preliminary transformation of an initial system to amplitude-phase variables and also becomes complicated in the case of bifurcations (multiple resonance).[19,20]

The most natural approach for analysis of Eq. (1) in the case of bifurcations is the Poincaré normal forms method which has become an essential tool in modern qualitative bifurcation theory.[19,20] The Poincaré method acts in the cases when $f(x = 0, \cdot, \cdot, \cdot) = 0$ and f is a power series. For such a system, a normal forms method specifies the simplest canonical form which preserves the stability characteristics of an initial plant. The normal form is deduced from the original system by nonlinear change of variables adapted to the structure of the system. The normalized change of variables is addressed as a recursively defined power series which is poorly computed.[19,20] Although the Poincaré method has been used widely for topological classification of bifurcations, its practical applications are limited due to a lack of computability.

Generalized Normal Forms Method and Averaging Normal Forms

We briefly describe here a more general approach to finding normal forms which is computationally efficient and is applicable to a broader class of systems than established algorithms. This approach employs useful features of the previously mentioned basic techniques. Write an equation in the form

$$\dot{x} = Ax + \epsilon f(x, t, \epsilon) \tag{2}$$

where f is a power series in x periodic with respect to t and A is as in Eq. (1). Note that we do not have to assume here that $f(0, t, \epsilon) = 0$. As earlier, we attempt to simplify Eq. (2) with the aid of a close-to-identity change of variables

$$x = y + \epsilon r(y, t, \epsilon)$$

Suppose that in the new variables Eq. (2) takes the form

$$\dot{y} = Ay + \epsilon B(y, t, \epsilon) \tag{3}$$

From Eqs. (2) and (3) we get

$$Ar - r_{,y}Ay - r_{,t} = -f(y + \epsilon r) + B + \epsilon r_{,y}B \tag{4}$$

where $r_{,y}$ is the Jacobian with respect to y and $r_{,t}$ is a partial derivative of r with respect to t. We consider Eq. (4) as a system of partial differential equations with respect to r. We seek a particular solution of Eq. (4) satisfying an additional condition

$$\|f\| \to 0 \Rightarrow \|r\| \to 0$$

where $\|\cdot\|$ is a certain norm of a vector-valued function. In other words, we are looking for a particular solution such that the norm of r tends to zero if the norm of f tends to zero also. For small ϵ one can try to approximate the solution of Eq. (4) by iterations

$$\begin{aligned} Ar_1 - r_{1,y}Ay - r_{1,t} &= -f(y, t, \epsilon) \\ Ar_k - r_{k,y}Ay - r_{k,t}\ eq &-f(y + \epsilon r_{k-1}, t, \epsilon) \\ &+ B_k + \epsilon r_{k-1,y}B_{k-1}, \quad k > 1 \end{aligned} \tag{5}$$

Note that Eq. (5) represents a set of linear partial differential equations with respect to r_k. Let us write a characteristic equation for Eq. (5)

$$\dot{r}_1 = Ar_1 + f(y, t, \epsilon) - B_1 \tag{6}$$

$$\dot{r}_k = Ar_k + f(y + \epsilon r_{k-1}, t, \epsilon) - \epsilon r_{k-1,y}B_{k-1} - B_k, \quad k > 1 \tag{7}$$

$$\dot{y} = Ay \tag{8}$$

It follows from Eq. (8) that $y = e^{At}c$, where c is a constant vector.

We split the time-dependent terms in Eqs. (6–8) into two distinct groups, namely, resonance and nonresonance terms.

Definition: $F(t)$ is called a resonance perturbation if

$$F(t) = e^{At}N(c)$$

where $N(c) \in R^n$ is a vector dependent on c, otherwise F is called a nonresonance perturbation. To annihilate nonresonance terms we set $B_k = e^{At}N_k(c)$. Recalling that $c = e^{-At}y$, one is able to write

$$B_k(y, t) = e^{At}N_k(e^{-At}y)$$

It is easy to verify that the last formula generalizes the Poincaré resonance condition and agrees with it if f admits Poincaré's assumptions. Note also that B_k becomes time invariant if f does not include time.

Observe that Eqs. (6–8) coincide with the set of linear ordinary differential equations given by naive iterations if one sets $B_k = 0$, $k \geq 1$. As mentioned before, the resonance terms present in these recursive sequences force the iterations to diverge on large time intervals. The problem with naive iterations is that they do not distinguish between resonance and nonresonance terms. In contrast with this, one can use the available B_k to annihilate resonance terms in Eqs. (6–8) and place them into the normal form (3).

Because the resonance terms are exceptional, the normalization yields significant reduction of the initial nonlinear system. Moreover, the normal forms admit complementary reduction in the amplitude-phase variables.

Let us write the normal form equation in such a way,

$$\dot{y} = Ay + \epsilon e^{At}N(e^{-At}y, \epsilon) + O(\epsilon^k)$$

Addressing a slow variable $c(t)$ by the formula $y(t) = e^{At}c(t)$, one obtains

$$\dot{c} = \epsilon N(c, \epsilon) + O(\epsilon^k)$$

Assume now that all eigenvalues of A are complex conjugate

$$\lambda_k = \alpha_k + I\omega_k; \qquad \bar{\lambda}_k = \alpha_k - I\omega_k,$$

$$k = 1, \ldots, n/2, \qquad I = \sqrt{-1}$$

Let us represent a complex diagonal matrix $A = \alpha + I\beta$ and introduce $y = e^{I\beta t}c(t)$, where related c_k are chosen as complex conjugate couples

$$c_k(t) = a_k(t)e^{I\rho_k(t)}, \qquad \bar{c}_k(t) = a_k(t)e^{I\rho_k(t)}, \qquad k = 1, \ldots, n/2$$

Assume also, that vector $\lambda = \{\lambda_1, \ldots, \lambda_{n/2}\}$ is not resonant. Namely, there is no vector m satisfying the Poincaré resonance condition $\lambda_s = (m;\lambda)$, $s = 1, \ldots, n/2$. In this case we get

$$\begin{aligned} \dot{a} &= \alpha a + \tfrac{1}{2}\epsilon\,[N(a, \epsilon) + \bar{N}(a, \epsilon)] + O(\epsilon^k) \\ \dot{\rho} &= -\,[(-I)/2]a^{-1}\epsilon[N(a, \epsilon) - \bar{N}(a, \epsilon)] + O(\epsilon^k) \end{aligned} \tag{9}$$

where $a = \{a_1, \ldots, a_{n/2}\}$, $\rho = \{\rho_1, \ldots, \rho_{n/2}\}$, and $\alpha = \{\alpha_1, \ldots, \alpha_{n/2}\}$. Note that the remarkable feature of Eqs. (9) is that the right-hand sides are independent of ρ and also $a \geq 0$.

For zero eigenvalues, related eigenvectors occur as slow variables directly. Assume that critical eigenvalues consist of both a number of zero eigenvalues and a certain number of noncomensurable complex conjugate couples. In this case an averaging normal form also assumes the form (9). In fact, the vector of slow variables consists of additional components corresponding to zero eigenvectors.

In the cases when additional resonances occur, $\lambda_s = (m; \lambda)$, $s = 1, \ldots, p$ conforming combinations of phase variables, namely,

$$\psi_s = (m;\rho) - \rho_s, \qquad s = 1, \ldots, p \tag{10}$$

play the same part as additional amplitude variables. Defining by Eq. (10) the new variables $\psi = \{\psi_1, \ldots, \psi_p\}$ and leaving the same notation for the remaining variables, we write Eq. (9) in a modified form

$$\dot{a} = \alpha a + \tfrac{1}{2}\epsilon\,[N(a, \psi, \epsilon) + \bar{N}(a, \psi, \epsilon)] + O(\epsilon^k)$$

$$\dot{\psi} = -\,[(-I)/2]a^{-1}\epsilon[N(a, \psi, \epsilon) - \bar{N}(a, \psi, \epsilon)] + O(\epsilon^k)$$

$$\dot{\rho} = [(-I)/2]a^{-1}\epsilon[(N(a, \psi, \epsilon) - \bar{N}(a, \psi, \epsilon)] + O(\epsilon^k)$$

Thus, additional p resonances yield additional p active variables (ψ) in the averaging normal form.

In Appendix A we present upper bounds on residual terms which are used in the next section in the design of a stabilizing control.

Resonance Stabilization

We outline briefly the essential steps in design of the resonance control. Assume that a plant is represented by equations of the averaging normal form

$$\dot{a} = \alpha a + \epsilon Q(a, \epsilon) + \epsilon^k E(a, \rho, \epsilon) + V$$

$$\dot{\rho} = \epsilon q(a, \epsilon) + \epsilon^k e(a, \rho, \epsilon) + v$$

where uncertain values v and V ($v_{\min} < v < v_{\max}$, $V_{\min} < V < V_{\max}$) initialize bounded uncertainties and E and e are residuals.

It is assumed that possible error involved in an approximation of the steady-state solution has been estimated and has been formally added to the bounds of unmodeled dynamics. We also assume that the origin of coordinates is shifted to coincide with steady-state solutions. In Lyapunov's asymptotic stability analysis, the phase variables are not essential and the relevant phase equations can be discarded.

Assign

$$G(a) = \alpha a + \epsilon Q(a, \epsilon) + \epsilon^k E(a, \rho, \epsilon) + V + u(a)$$

$$G^0(a) = \alpha a + \epsilon Q(a, \epsilon) + \epsilon^k E^0(a, \epsilon) + V + u(a)$$

where feedback $u(a)$ is addressed in amplitude variables.

Choose a Lyapunov function $L = L(a)$ and design feedback to satisfy in a certain neighborhood of a steady-state solution the Lyapunov criteria for asymptotic stability

$$\sum_{i=1}^{n_1} \frac{\partial L}{\partial a_i} G_i(a, \rho, \epsilon) \leq -d, \qquad \mu \leq |a| \leq a^0, \qquad \mu > 0$$

$$\sum_{i=1}^{n_1} \frac{\partial L}{\partial a_i} G_i(a, \rho, \epsilon) \leq 0, \qquad |a| < \mu$$

where vector $D > 0$ is assigned the degree of relative stability, vector and a^0 initializes a basin of stability, and μ is a small positive value. Note that in the last relation, equality is achieved only for $a = 0$. The preceding condition consists of ρ and thus would be difficult to verify. Recalling that $\partial L/\partial a_i > 0$, we are able to simplify this inequality. In fact, using the upper bound on the residual we get

$$\sum_{i=1}^{n_1} \frac{\partial L}{\partial a_i} G_i^0(a, \epsilon) \leq -d, \qquad \mu \leq |a| \leq a^0, \qquad \mu > 0$$

$$\sum_{i=1}^{n_1} \frac{\partial L}{\partial a_i} G_i^0(a, \epsilon) \leq 0, \qquad |a| < \mu \tag{11}$$

There are a few ways to determine a feedback $u(a)$ that satisfies the Eq. (11) inequality. A simple control design procedure is addressed in the next two examples.

The final step is to map feedback $u(a)$ from the amplitude to the original state variables. This step is rather routine because the truncated averaging normal form is linked with the initial equations by an invertible mapping given in closed form.

In Appendix B we present clear and physically meaningful stability conditions which are employed next in the design of a resonance stabilizing feedback control.

Resonance Control of the Hopf Bifurcation

The Van der Pol equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(x_2 - x_2^3)$$

is one of the simplest exhibiting Hopf bifurcation. The eigenvalues of the linearized system ($\lambda_1 = I$, $\lambda_2 = -I$) are a pure imaginary couple and satisfy the resonance condition $\lambda_1 + \lambda_2 = 0$. The resonance condition is violated for any small $\epsilon \neq 0$ and occurs suddenly in the $\epsilon = 0$ case. Indeed, the normal forms in the resonance and nonresonance cases are essentially distinct. To find a normal form which smoothly depends on the parameter ϵ, one has to treat the small linear term ϵx_2 as part of the function f. Thus, the associated linearized system is

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1$$

The first-order approximation of the resonance normal form for $\epsilon = 0.1$ is

$$\dot{z}_1 = 0.05z_1 - 0.15z_1^2z_2 + O(\epsilon^2)$$
$$\dot{z}_2 = -0.05z_2 - 0.15z_1z_2^2 + O(\epsilon^2)$$

The variables $\{z_1, z_2\}$ are coupled with $\{x_1, x_2\}$ by the nonlinear change of variables

$$x = Ty$$
$$y_1 = z_1 - 0.025Iy_2 - 0.025Iy_1^3 + 0.075Iy_1y_2^2 - 0.0125Iy_2^3$$
$$y_2 = z_2 + 0.025Iy_1 + 0.025Iy_2^3 - 0.075Iy_1^2y_2 + 0.0125Iy_1^3$$

In amplitude-phase variables $\{a, \rho\}$

$$z_1 = a(t)e^{It + I\rho(t)} \qquad z_2 = a(t)e^{-It - I\rho(t)}$$

the normal form is represented as follows:

$$\dot{a} = 0.05a - 0.15a^3 + \epsilon^2E(a, \rho)$$
$$\dot{\rho} = \epsilon^2 e(a, \rho)$$

where residuals $E(a, \rho)$ and $e(a, \rho)$ are determined in the second iteration.

It is clear that the trivial solution is unstable in this case, while the system possesses a stable limit circle. To stabilize the trivial solution of the given system by feedback, we determine the upper bound of $E(a, \rho)$

$$E(a, \rho) < E^0 = 0.00928a^3 + 0.034599a^5 + 0.00253a^7 + 0.0002a^9$$

Combining the resonance and error terms one gets

$$g(a) = 0.05a - 0.14072a^3 + 0.034599a^5 + 0.00253a^7 + 0.0002a^9$$

Observe that a constant value is not a resonant component, and thus both $u(0) = 0$ and $g(0) = 0$. We choose a control $u(a)$ which ensures that

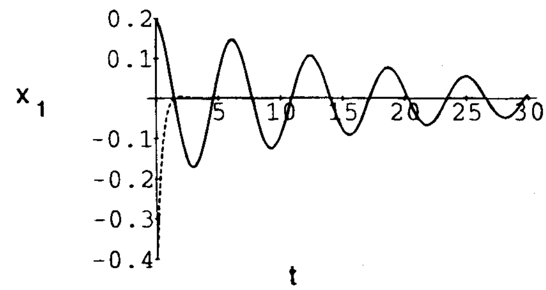

Fig. 1a Combined graph of time history of controlled Van der Pol equations (solid line) and control force required for stabilization (dashed line).

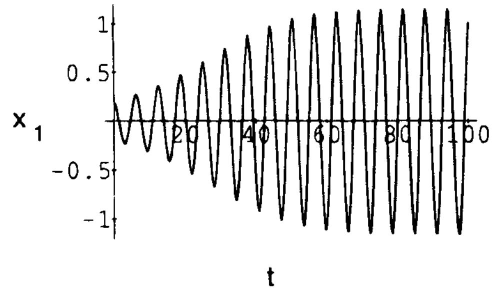

Fig. 1b Runge-Kutta solution of the Van der Pol equations.

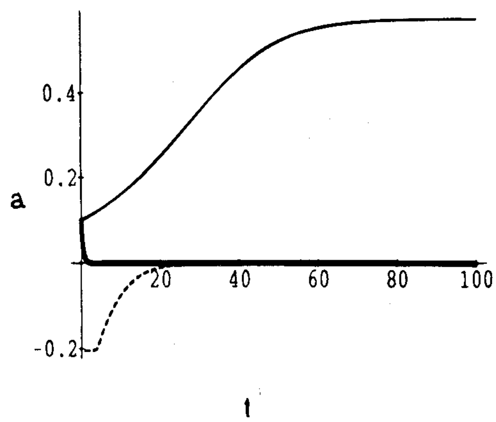

Fig. 1c Combined graph of uncontrolled amplitude (solid line), controlled amplitude (thick line), and control force (dashed line).

$$G^0(a) = g(a) + u(a) \leq -d, \quad \text{for } a > \mu$$
$$G^0(a) \leq 0, \quad \text{for } 0 \leq a < \mu \tag{12}$$

In other words, control should ensure asymptotic stability for all $a \geq 0$, and outside of a small region about zero (initialized by vector μ), the least rate of decay of the amplitude variable should be equal to d. Choose control

$$u(a) = ka - 0.035a^5 - 0.0026a^7 - 0.00021a^9$$

to overcome small positive nonlinear terms in $g(a)$; at the same time the prime resonance nonlinear term remains the same.

Let $\mu = 0.005$ and $d = 0.1$. Then determining k that satisfies Eq. (12), we get

$$u(a) = -2.04965a - 0.035a^5 - 0.0026a^7 - 0.00021a^9$$

Mapping this function back to the original variables introduces control in measurable initial coordinates. Figure 1b shows the initial system with unstable equilibrium and Figure 1a displays the effect of the designed stabilizing control. On Figure 1c we display both uncontrolled and controlled amplitudes as well as stabilizing feedback vs time.

Analysis and Control of an Aircraft Model

In this section we analyze and control the lateral oscillations experienced by a slender-wing aircraft. The considered model was adopted from Cochran and Ho[3] which they termed aircraft model B. The obtained averaging normal form reveals the failure of linearization in critical cases.

$$A = \begin{bmatrix} \frac{\rho SVC_{yl}}{m} & \frac{g\cos(\alpha)}{V} & \sin(\alpha)+\frac{\rho SsC_{y_p}}{m} & -\cos(\alpha)+\frac{\rho SsC_{y_r}}{m} \\ 0 & 0 & 1 & \tan(\alpha) \\ q_xV(I_zC_{t_l}+I_{xz}C_{nl}) & 0 & q_xs(I_zC_{t_p})+I_{xz}C_{n_p}) & q_xs(I_zC_{t_r}+I_{xz}C_{n_r}) \\ q_xV(I_{xx}C_{t_l}+I_xC_{nl}) & 0 & q_xs(I_{xz}C_{t_p}+I_xC_{n_p}) & q_xs(I_{xz}C_{t_r}+I_xC_{n_r}) \end{bmatrix}$$

$$f = \begin{bmatrix} \frac{(\rho bSV)C_{y_3}\beta^3}{2m} \\ 0 \\ q_xV(I_zC_{t_3}+I_{xz}C_{n_3})\beta^3 \\ q_xV(I_{xz}C_{t_3}+I_xC_{n_3})\beta^3 \end{bmatrix}$$

The governing equations are written in the form (1) where with β the sideslip angle, ϕ the roll angle, p the roll rate, and r the yaw rate. The value of listed parameters were taken from Table 2 in Cochran and Ho[3] with the exception of $I_{xx} = 2182$ kg/m^2, $C_{t3} = -2.0$, along with angle of attack $\alpha = 1.25$ deg, and speed $V = 71.25$ m/s.

With these parameters the equations of motion take the form

$$\begin{aligned} \dot{x}_1 &= -0.126927x_1 - 1.98427x_1^3 + 0.0432864x_2 \\ &\quad + 0.947983x_3 - 0.314497x_4 \\ \dot{x}_2 &= x_3 + 3.01834x_4 \\ \dot{x}_3 &= -16.9354x_1 - 161.428x_1^3 - 0.358942x_3 \\ &\quad + 1.04395x_4 \\ \dot{x}_4 &= -5.16107x_1 - 25.4932x_1^3 - 0.212874x_3 \\ &\quad + 0.00577748x_4 \end{aligned} \tag{13}$$

where $x = (\beta\ \phi\ p\ r)^T$. We will represent a stabilizing control as a polynomial of the state variables at the end of this section.

The eigenvalues of the system (13) linearized are

$$\lambda = \{-0.00181506 + 3.83335\,I, -0.00181506 - 3.83335\,I, -0.47646, -8.61259\ 10^{-7}\}$$

Notice that both the real portion of the complex conjugate pair and the last eigenvalue in the preceding list are close to zero. These relevant small linear terms in Eq. (13) are adjoint to f such that the principal linear part in Eq. (13) becomes critical. As was mentioned, the normal form of such a critical system preserves behavior of the initial system in a sufficiently large neighborhood of the steady-state solution.

Calculation of the first-order normal form for this system gives

$$\begin{aligned} \dot{z}_1 &= (-0.00181506 + 3.83335\,I)z_1 + (-0.308908 \\ &\quad + 3.11451\,I)z_1^2z_2 + (-0.000967355 \\ &\quad + 0.00975318\,I)z_1z_4^2 \\ \dot{z}_2 &= (-0.00181506 - 3.83335\,I)z_2 + (-0.308908 \\ &\quad - 3.11451\,I)z_1z_2^2 + (-0.000967355 \\ &\quad - 0.00975318\,I)z_2z_4^2 \\ \dot{z}_3 &= -0.47646z_3 + 0.474638z_1z_2z_3 + 0.000743172z_3z_4^2 \\ \dot{z}_4 &= -8.61259\ 10^{-7}z_4 + 0.103323z_1z_2z_4 + 5.39267\ 10^{-5}z_4^3 \end{aligned}$$

The variable z is determined by a cumbersome nonlinear transformation which is obtained with the aid of symbolic software.

Addressing amplitude-phase variables by the formula

$$\begin{aligned} z_1 &= a(t)\exp\{I[3.83335t + \rho(t)]\} \\ z_2 &= a(t)\exp\{-I[3.83335t + \rho(t)]\} \\ z_3 &= z_3; \qquad z_4 = z_4 \end{aligned}$$

we get

$$\begin{aligned} \dot{a} &= -0.00181506a - 0.308908a^3 - 0.000967355az_4^2 \\ \dot{\rho} &= 3.11451a^2 + 0.00975318z_4^2 \\ \dot{z}_3 &= -0.47646z_3 + 0.474638a^2z_3 + 0.000743172z_3z_4^2 \\ \dot{z}_4 &= -8.61259\ 10^{-7}z_4 + 0.103323a^2z_4 + 5.39267\ 10^{-5}z_4^3 \end{aligned} \tag{14}$$

The first and last equations denote projection of the system on the center manifold which characterizes stability properties of the system. Thus the remaining two equations can be disregarded in stability analysis.

Although Eq. (14) remains nonintegrable, it admits clear qualitative analysis. Numerical integration of Eq. (14) shows that amplitude a decays slowly and z_4 grows slowly.

To estimate the error involved in the normal form reduction, we map these solutions to the original coordinate basis (see Figs. 2a–2d) and compare them with direct numerical simulations of the initial equations. In fact, on the displayed time interval the error remains less than 10^{-3}. Smooth nonlinear feedback control that ensures exponential decay of a and z_4 coordinates with an assigned rate of convergence to zero can be introduced as a function of only two critical variables, a and z_4.

Computing the second iteration, we find the error residual of Eq. (14) and calculate an upper bound on the residual. Recall that the upper bound is a strictly increasing function and so reaches its maximum value at the boundary of the stability region: $0 \leq a < a^0$, $|z_4| \leq z_4^0$. Thus,

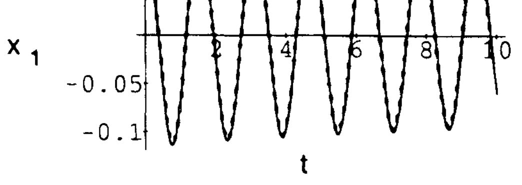

Fig. 2a Comparison of Runge-Kutta (solid line) and analytical solution (dashed line) of x_1 for model B aircraft.

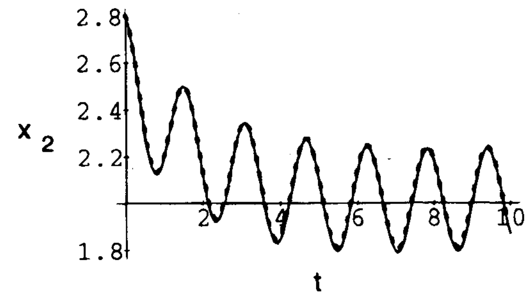

Fig. 2b Comparison of Runge-Kutta (solid line) and analytical solution (dashed line) of x_2 for model B aircraft.

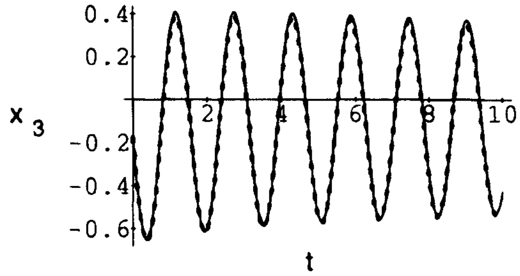

Fig. 2c Comparison of Runge-Kutta (solid line) and analytical solution (dashed line) of x_3 for model B aircraft.

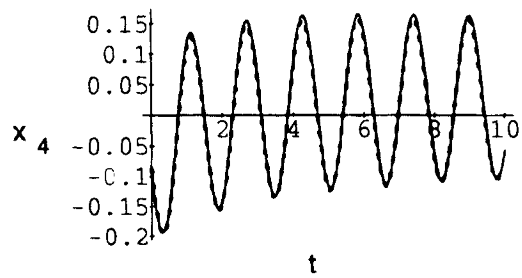

Fig. 2d Comparison of Runge-Kutta (solid line) and analytical solution (dashed line) of x_4 for model B aircraft.

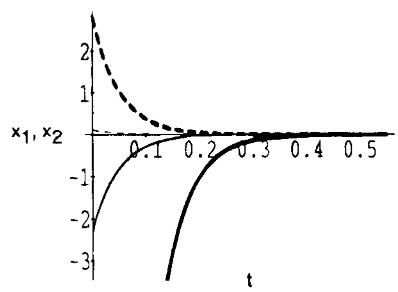

Fig. 2e Combined graph of time history of controlled model B equations (x_1 thin dash, x_2 thick dash) and control force required for stabilization (x_1 thin, x_2 thick).

$$g_1 = -0.00181506a - 0.308908a^3 - 0.0000967355az_4^2 + \max[E_1(a^0, z_4^0)]$$

$$g_2 = 0.103323a^2z_4 + 5.39267\ 10^{-5}z_4^3 + \max[E_2(a^0, z_4^0)]$$

Control $u = \{u_1, u_2\}$ is chosen as follows:

$$u_1 = k_1 a$$

$$u_2 = k_2 a - 0.103323a^2z_4 - 5.39267\ 10^{-5}z_4^3$$

Note that in the first equation the resonance nonlinear terms enhance stability and so remain unchanged, whereas the nonlinear term in the last equation destabilizes the system and thus has been annihilated by feedback. The coefficients $k_1 = -20.0025$ and $k_2 = -20.0025$ are chosen to satisfy the stability conditions (B1) and (B2) (see Appendix B), with $d = 0.1$. The computation has been completed for

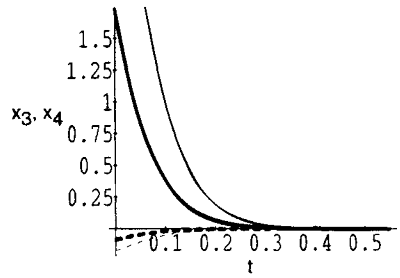

Fig. 2f Combined graph of time history of controlled model B equations (x_3 thin dash, x_4 thick dash) and control force required for stabilization (x_3 thin, x_4 thick).

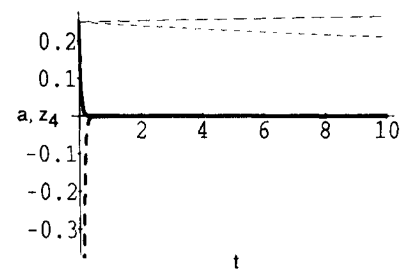

Fig. 2g Model B combined graph of time histories of uncontrolled amplitudes of unstable modes (a short dash, z_4 long dash), controlled amplitudes (single thick line), and control forces (single thick dashed line).

$$\mu_1 = 0.025, \qquad \mu_2 = 0.025, \qquad a^0 = 0.25, \qquad z_4^0 = 0.25$$

$$\max(E_1) = 2.17986\ 10^{-5}, \qquad \max(E_2) = 1.32859\ 10^{-7}$$

In initial coordinates, control $Q = \{Q_1, \ldots, Q_4\}$, $Q = Q(x_1, \ldots, x_4)$, appears as a polynomial of degree three with respect to all state variables. Since the accurate control description is cumbersome, we represent here only essential components in Q_1,

$$Q_1 \approx -19.994x_1 + 0.041x_1^2x_3 - 0.134x_1^2x_4 + \cdots$$

Figs. 2e and 2f display the result of resonance stabilization in the original coordinate frame. In Fig. 2g we display both uncontrolled and controlled amplitudes and stabilizing feedback vs time.

Conclusions

Recently, a number of observed instabilities were interpreted as bifurcations of steady state solutions of nonlinear dynamic models of aircraft. Although certain bifurcation phenomena of aircraft have been analyzed, there exists a clear need for a unified methodology providing analysis and efficient control of instabilities of multidimensional nonlinear plants modeling flight performance. In this paper, we have described a generic method which furnishes an accurate reduction of multidimensional aircraft models to further nonreducible resonance normal forms. The method is computationally efficient and has been realized in symbolic computer software, providing one with a low-dimensional nonlinear model preserving local stability properties of the initial plant. The separation of fast and slow modes (averaging) turns out to be trivial for the normal form equations and is given in a closed form. Although residuals of the recursive normal forms approximation are coupled with both fast and slow variables, we deduce sharp bounds, which depend only on slow variables. Using these bounds, we reveal the Lyapunov stability of a chosen steady-state solution in terms of a plant's slow variables.

A robust feedback stabilization of aircraft instabilities is ensured by a novel control design procedure linked with the method of stability analysis. We choose feedback as a polynomial function of slow variables and reach a required behavior by adjusting free feedback parameters. After that, feedback is mapped into initial measurable coordinates. Such a design procedure is efficient because it intrinsically voids the nonresonance portion of the system which does not affect stability. The developed methodology was applied to analysis and control of unstable behavior of a nonlinear model of the lateral dynamics of an aircraft. The analytical results were verified by numerically integrating the nonlinear state equations. It was shown that the method provides correct qualitative as well as quantitative results in the case of a complex bifurcation linked with a pair of complex conjugate and zero critical eigenvalues.

Note that relative complexity of the addressed methodology is essentially reduced by its computerization. Indeed, the prospects of having both complete stability analysis and efficient stabilizing control of complex bifurcation behavior make the computational effort worthwhile.

Appendix A: Residual Bounds

We present here a simple upper bound on residual terms that guides the formulation of sufficient conditions for local stability of a steady-state solution.

Terminating iterations (6–8) on the kth step by setting $r_{k+1} = r_k$ gives the residual term in the form $E = B_{k+1} = Ar_k - r_k + \gamma(y + \epsilon r_k) - \epsilon r_{k,y} B_k$. It is clear that $E = E(a, \rho, \epsilon)$ depends on both amplitude and phase variables. Therefore, Eq. (9) can be written in the following form:

$$\begin{aligned} \dot{a} &= \alpha + ½\, \epsilon[N(a, \epsilon) + \bar{N}(a, \epsilon)] + \epsilon^k E(a, \rho, \epsilon) \\ \dot{\rho} &= [(-I)/2\, a^{-1}\epsilon[N(a, \epsilon) - \bar{N}(a, \epsilon)] + \epsilon^k e(a, \rho, \epsilon) \end{aligned} \tag{A1}$$

where amplitudes $a \in R^{n_1}$ and phases $\rho \in R^{n_2}[(n_1 - n_2)$ is the number of zero eigenvalues in matrix $\alpha]$. E and e are periodic functions with respect to each ρ_k, $k = 1, \ldots, p$, with noncomensurable periods T_k

$$E(a, \rho_k + T_k) = E(a, \rho_k), \qquad e(a, \rho_k + T_k) = e(a, \rho_k)$$

It is important to notice that the preceding equations are given by the closed-form change of variables and, therefore, are accurate. Let us show that the residuals E and e can be effectively bounded.

Indeed, the bounds are

$$|E| \leq \max_{\rho_1 \in [0;\, T_1]} \cdots \max_{\rho_n \in [0;\, T_n]} E(a, \rho, \epsilon) = E^0(a, \epsilon)$$

$$|e| \leq \max_{\rho_1 \in [0;\, T_1]} \cdots \max_{\rho_n \in [0;\, T_n]} e(a, \rho, \epsilon) = e^0(a, \epsilon)$$

Using these bounds we can formulate asymptotic stability conditions for a steady-state solution of Eq. (A1) using a Lyapunov function which is addressed in amplitude variables. Observe that in the majority of engineering problems, the residual turns out to be insignificantly small and stability is governed by the resonance part of Eq. (A1).

Appendix B: Sufficient Stability Conditions

Although a variety of Lyapunov functions can be adapted in feedback design, we will show that the most simple one

$$L = \sum_{i=1}^{n_l} a_i^2$$

provides us with a controller which yields exponential decay of the amplitude variables with the desired rates. The amplitudes a_i are non-negative when all critical/close-to-critical eigenvalues are complex conjugate with nonzero imaginary parts and there are no additional resonances imposed.

In this case $a_0 = 0$ and Eq. (11) is true if

$$\begin{aligned} G_i^0(a) &\leq -d, & \mu \leq a \leq a^0 \\ G_i^0(a) &\leq 0, & 0 \leq a < \mu \end{aligned}$$

Rewrite this inequality in the form

$$\begin{aligned} \dot{a}_i &\leq -d_i, & \mu \leq a \leq a^0 \\ \dot{a}_i &\leq 0, & 0 \leq a < \mu \end{aligned} \tag{B1}$$

Thus, the resonance stabilization allows us to control least rates (d_i) of exponential decay of the amplitude variables which is one of the principal concerns in engineering applications.

Suppose now that all critical eigenvalues, are real numbers close or equal to zero. In this case Eq. (11) holds, but there is no reason to assume that $a_i > 0$. In particular, for a single zero eigenvalue, Eq. (11) is reduced to

$$\begin{aligned} aG^0(a) &\leq -d, & \mu \leq a \leq a^0 \\ aG^0(a) &\leq 0, & |a| < \mu \end{aligned}$$

To conform with this relation, $G(a)$ should possess the property

$$\text{sign}\, G(a) = -\,\text{sign}\, a$$

The last relation agrees if $G(a)$ is chosen for example as an appropriate smooth odd function.

In the case of multiple zero/close-to-zero eigenvalues, Eq. (11) is reduced to

$$\begin{aligned} \sum_{i=1}^{n_1 - n_2} a_i G_i^0(a) &\leq -d, & \mu \leq a \leq a^0 \\ \sum_{i=1}^{n_{1n_2}} a_i G_i^0(a) &\leq 0, & |a| < \mu \end{aligned} \tag{B2}$$

This relation agrees if

$$\begin{aligned} &\text{sign}\, G_i^0(a) = -\,\text{sign}\, a_i, \qquad G_i^0(a = 0) = 0, \\ &|G_i^0(a)| < d_i, \qquad a_0 \leq a \leq -\mu, \qquad \mu \leq a \leq a^0 \end{aligned}$$

where rates of decay d_i are determined from Eq. (B2).

The general case when both complex and real critical eigenvalues persist can be broken out into the two special cases mentioned earlier.

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